Riemann Integral

Let $I = [a,b]$ be closed and bounded interval.
Then a finite set of real numbers

$$P = \{x_0, x_1, x_2, \dots x_n\}$$

having property that $a = x_0 < x_1 < \dots < x_n = b$

is called partition of closed interval $[a,b]$

Closed sub intervals

$$I_1 = [x_0, x_1], \quad I_2 = [x_1, x_2], \quad \dots \quad I_n = [x_{n-1}, x_n]$$

are called segments of partition
Δx_r denote length of r^{th} sub interval

$$\Delta x_r = x_r - x_{r-1}$$

✓ The norm of partition P is maximum of length of segment of partition P & is denoted by $\|P\|$

$$\|P\| = \max(\Delta x_r : r = 1, 2, \dots n)$$

Refinement :- Partition P^* is called refinement of another partition P iff $P^* \supset P$ i.e. every point of P is used in constructing P^*.

Lower Riemann Sum, Upper Riemann Sum (Darboux sum)

Let f be bounded real valued function defined on bounded and closed interval $[a,b]$ and let $P = \{a = x_0, x_1, \dots x_n = b\}$ be partition of $[a,b]$

Let m_r & M_r be infimum and supremum of function f on sub interval $[x_{r-1}, x_r]$, then

$$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r$$

and

$$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r$$

are called lower Riemann sum and upper Riemann sum of f on $[a, b]$ w.r.t partition P.

$$U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r) \Delta x_r$$

Then $\sum_{r=1}^{n} (M_r - m_r) \Delta x_r$ is called Oscillatory sum for the sum function f.

Some properties of Darboux sum

Prop. I If P is a partition of interval $[a, b]$ and f is bounded function defined on $[a, b]$ then

$$M(b-a) \geq U(P, f) \geq L(P, f) \geq m(b-a)$$

where $M = \sup f$, $m = \inf f$

Proof: Let $P = \{a = x_0, x_1, \ldots x_n = b\}$ be any partition of $[a, b]$ & let $I_r = [x_{r-1}, x_r]$, $r = 1, 2, \ldots n$ be subintervals of $[a, b]$. Let m_r and M_r be inf and sup of f in $[x_{r-1}, x_r]$

$$M \geq M_r \geq m_r \geq m$$

or $M \Delta x_r \geq M_r \Delta x_r \geq m_r \Delta x_r \geq m \Delta x_r$

or $\sum_{r=1}^{n} M \Delta x_r \geq \sum_{r=1}^{n} M_r \Delta x_r \geq \sum_{r=1}^{n} m_r \Delta x_r \geq \sum_{r=1}^{n} m \Delta x_r$

or $M(b-a) \geq U(P,f) \geq L(P,f) \geq m(b-a)$

<u>Prop. II</u> If f be real valued bounded function def. on $[a,b]$ and if $P, P' \in [a,b]$ s.t. P' is refinement of P then

(i) $L(f,P) \leq L(f,P')$

(ii) $U(f,P') \leq U(f,P)$

<u>Proof</u> Let $P = \{a = x_0, x_1, \ldots x_n = b\}$ and

$P' = \{a = x_0, x_1, \ldots, x_{r-1}, \alpha, x_r, \ldots x_n = b\}$

Let m_r, m_r' & m_r'' be infimum of subintervals $[x_{r-1}, x_r]$, $[x_{r-1}, \alpha]$ & $[\alpha, x_r]$

So $m_r \leq m_r'$ & $m_r \leq m_r''$

$m_r'(\alpha - x_{r-1}) + m_r''(x_r - \alpha) \geq m_r(\alpha - x_{r-1}) + m_r(x_r - \alpha)$

$\geq m_r(x_r - x_{r-1})$

$\Rightarrow$ $L(f, P') \geq L(f, P)$ — (1)

also supremum of $[x_{r-1}, x_r]$ $[x_{r-1}, \alpha]$ $[\alpha, x_r]$ be M_r, M_r' & M_r'', $M_r \geq M_r'$ & $M_r \geq M_r''$

we can prove $U(f, P') \leq U(f, P)$ — (2)

from (1) & (2) $L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,P)$

Prop III If $P_1, P_2 \in P[a,b]$ then

i) $L(f, P_1) \leq U(f, P_2)$

ii) $L(f, P_2) \leq U(f, P_1)$

i.e. Lower Darboux sum never exceed any of the upper Darboux sum.

Proof Let $P_1 \cup P_2$ is common refinement of P_1 & P_2

$P_1 \subset P_1 \cup P_2$ & $P_2 \subset P_1 \cup P_2$

Since $P_1 \cup P_2$ is refinement of P_1

$$L(f, P_1) \leq L(f, P_1 \cup P_2) \quad -(1)$$

$$\& \quad U(f, P_1 \cup P_2) \leq U(f, P_1) \quad -(2)$$

from (1) & (2) $L(f, P_1) \leq L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) \leq U(f, P_1)$ — (3)

also $P_1 \cup P_2$ is refinement of P_2

$L(f, P_2) \leq L(f, P_1 \cup P_2)$ and $U(f, P_1 \cup P_2) \leq U(f, P_2)$

So $L(f, P_2) \leq L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) \leq U(f, P_2)$ — (4)

from (3) & (4) $L(f, P_1) \leq U(f, P_2)$

and $L(f, P_2) \leq U(f, P_1)$

Prop IV :- If P is any partition of interval $[a,b]$ and f and g are bounded function defined on $[a,b]$ then

$$U(P, f+g) \leq U(P, f) + U(P, g)$$

and $L(P, f+g) \geq L(P, f) + L(P, g)$

Proof

Let $P = \{a = x_0, x_1, \dots x_n = b\}$ be any partition of $[a,b]$. Let f & g be two bounded functions defined on $[a,b]$ then $f+g$ is also bounded on $[a,b]$.

Let m_r', m_r'' & m_r be infimum value of f, g & $f+g$
M_r', M_r'' & M_r be supremum of f, g and $f+g$
we have

$$f(x) \geq m_r', \quad g(x) \geq m_r'' \quad \forall\, x \in [x_{r-1}, x_r]$$

$$\Rightarrow \quad f(x) + g(x) \geq m_r' + m_r'' \quad \forall\, x \in [x_{r-1}, x_r]$$

$$\Rightarrow \quad (f+g)x \geq m_r' + m_r'' \quad \forall\, x \in [x_{r-1}, x_r]$$

$\Rightarrow$ $m_r' + m_r''$ is lower bound of $f+g$ on $[x_{r-1}, x_r]$

now m_r is infimum of $f+g$
(g.l.b.)

Therefore $\quad m_r \geq m_r' + m_r''$

$$\Rightarrow \quad m_r \Delta x_r \geq m_r' \Delta x_r + m_r'' \Delta x_r$$

$$\Rightarrow \quad \sum_{r=1}^{n} m_r \Delta x_r \geq \sum_{r=1}^{n} m_r' \Delta x_r + \sum_{r=1}^{n} m_r'' \Delta x_r$$

$$\Rightarrow \quad L(P, f+g) \geq L(P, f) + L(P, g)$$

we can also prove

$$U(P, f+g) \leq U(P, f) + U(P, g)$$

Lower and Upper Riemann Integrals

Let f be bounded function on interval $[a,b]$ then lower ~~Rei~~ Riemann integral is defined as

$$\underline{\int_a^b} f(x)\,dx = \sup\{L(f,P)\}$$ and

upper Riemann integral is defined as

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(f,P)\}$$

Th. For a bounded function, the lower R integral cannot exceed upper R integral.

Proof Let $P_1, P_2 \in P[a,b]$ then

$$L(f,P_1) \le U(f,P_2)$$

keeping P_1 fixed & taking infimum over P_2 of $[a,b]$

$$L(f,P_1) \le \overline{\int_a^b} f(x)dx \text{ (by definition)}$$

taking Supremum over P_1

$$\Rightarrow \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx$$

✓ A bounded function f is said to be Riemann integrable over $[a,b]$ if its upper & lower Riemann integrals are equal.

i.e. $$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$$

✓ Every bounded function need not be R integrable.

Ex. $f(x) = \begin{cases} 0, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$

Since $f(x)$ is bounded in $[0,1]$

Let partition in $[0,1]$ be

$P = \{0 = x_0, x_1, \ldots x_n = 1\}$, $I_r = [x_{r-1}, x_r]$, $r = 1, 2, \ldots n$

Contain rational & irrational no.

for subinterval $[x_{r-1}, x_r]$

$M_r = 1$ & $m_r = 0$

$U(f, P) = \sum_{r=1}^{n} M_r (x_r - x_{r-1}) = \sum_{r=1}^{n} 1 \cdot (x_r - x_{r-1}) = 1$

$L(f, P) = \sum_{r=1}^{n} m_r (x_r - x_{r-1}) = 0$

$\overline{\int_0^1} f(x)dx = \inf U(f, P) = 1$ —(1)

$\underline{\int_0^1} f(x)dx = \sup L(f, P) = 0$ —(2)

From (1) & (2)

$\overline{\int_0^1} f(x)dx \neq \underline{\int_0^1} f(x)dx \Rightarrow f \notin R[0,1]$

$U(f,P) < \int_a^{\bar{b}} f(x)\,dx + \varepsilon$ (Darboux Theorem)

and $L(f,P) > \int_{\bar{a}}^{b} f(x)\,dx - \varepsilon$

<u>Th.</u> ~~Let~~ f be bounded function defined on $[a,b]$ then $f \in R[a,b]$ iff given $\varepsilon > 0$, there exist partition $P \in P[a,b]$ s.t.

$$U(f,P) - L(f,P) < \varepsilon$$

<u>Proof.</u> The condition is necessary

$$f \in R[a,b] \Rightarrow \int_a^{\bar{b}} f(x)\,dx = \int_{\bar{a}}^{b} f(x)\,dx = \int_a^b f(x)\,dx \quad (1)$$

from Darboux Theorem

$$U(f,P) < \int_a^{\bar{b}} f(x)\,dx + \frac{\varepsilon}{2} \quad \text{—(2)}$$

$$\& \; L(f,P) > \int_{\bar{a}}^{b} f(x)\,dx - \frac{\varepsilon}{2} \quad \text{—(3)}$$

Using (1) & (2) & (3)

$$U(f,P) < \int_a^b f(x)\,dx + \frac{\varepsilon}{2} \quad \text{—(4)} \qquad \& \qquad L(f,P) > \int_a^b f(x)\,dx - \frac{\varepsilon}{2}$$

$$\int_a^b f(x)\,dx < L(f,P) + \frac{\varepsilon}{2} \quad \text{—(5)}$$

adding (4) & (5)

$$U(f,P) + \cancel{\int_a^b f(x)\,dx} < \cancel{\int_a^b f(x)\,dx} + L(f,P) + \varepsilon$$

$$\Rightarrow U(f,P) - L(f,P) < \varepsilon$$

The condition is sufficient

The condition is sufficient

Let $\varepsilon > 0$, $P \in P[a,b]$ s.t.

$$U(f,P) - L(f,P) < \varepsilon$$

$$\overline{\int_a^b} f(x)\,dx = \operatorname{Inf} U(f,P) \Rightarrow \overline{\int_a^b} f(x)\,dx \leq U(f,P)$$

and $\underline{\int_a^b} f(x)\,dx = \sup L(f,P) \Rightarrow \underline{\int_a^b} f(x)\,dx \geq L(f,P)$

$$\Rightarrow -\underline{\int_a^b} f(x)\,dx \leq -L(f,P)$$

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq U(f,P) - L(f,P) < \varepsilon$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx < \underline{\int_a^b} f(x)\,dx + \varepsilon$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx \leq \underline{\int_a^b} f(x)\,dx$$

but $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$

hence $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx \Rightarrow f \in R[a,b]$